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# GENERALIZED WAVE EQUATIONS IN THE SETTING OF BESSEL-KINGMAN HYPERGROUPS

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*Dedicated to Professor Paul L. Butzer*
*on the occasion of his 80th anniversary*

**Abstract**

In this paper we investigate generalized Bessel potential operators in the setting of Bessel-Kingman hypergroups, and certain results using the Fourier-Bessel transform to be applied further for the regularities of the wave equations solutions.

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*Key Words and Phrases*: Fourier-Bessel transform, Sobolev imbedding, Bessel potential, regularity of solutions

## Introduction

The theory of Bessel potential operators has been introduced in ([1], p.220) on $\mathbb{R}^n$ using the classical Fourier transform. In this work, we investigate their analogues in the setting of the Bessel-Kingman hypergroups $\mathbb{K} = [0, \infty)$, denoted by $B_{\alpha,s}$ ($s \in \mathbb{R}$ , $\alpha > -\frac{1}{2}$). We are using the Fourier Bessel transform $\mathcal{F}_B$, where for a suitable function $f$ ([11], p.15)

$$\mathcal{F}_B f(\lambda) = \int_0^\infty j_\alpha(\lambda.) f dm_\alpha, \qquad \text{for all } \lambda \in \mathbb{R}, \tag{1}$$

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$dm_\alpha(x) = \dfrac{x^{2\alpha+1}dx}{2^\alpha\Gamma(\alpha+1)}$ and $j_\alpha$ is the modified Bessel function [12].

The Fourier-Bessel transform is an isomorphism from $S_*(\mathbb{R})$ into itself ([11] p.127), where $S_*(\mathbb{R})$ is the space of $\mathcal{C}^\infty$ even functions on $\mathbb{R}$, rapidly decreasing together with all their derivatives. Its inverse is given by $\mathcal{F}_B^{-1} = \mathcal{F}_B$ ([11], p.128).

In the present work we investigate the generalized Bessel potential operators $B_{\alpha,s} = \mathcal{F}_B^{-1}\left[(1+\xi^2)^{-\frac{s}{2}}\mathcal{F}_B(.)\right]$. These operators are well defined on $L^p(dm_\alpha)$, the weighted Lebesgue space corresponding to the measure $dm_\alpha$ and turning the space $W_\alpha^{s,p} = B_{\alpha,s}(L^p(dm_\alpha))$ into a Banach space with respect to the norm $||.||_{W_\alpha^{s,p}} = ||B_{\alpha,-s}(.)||_{L^p(dm_\alpha)}$. The Bessel potential operators are an important technical tool in Harmonic Analysis, Real Analysis, and Partial Differential Equations ([4],[9],[2]). Moreover, they are used here to investigate and describe the regularities of solutions for a generalized wave equations in the setting of hypergroups.

The paper is organized as follows: In the first section we collect some Harmonic Analysis results related to the Bessel operators. In the second section we study the Bessel potential operator $B_{\alpha,s}$ on the Bessel-Kingman hypergroups and give some properties and some imbedding results. Also, a Young inequality on $L^p(dm_\alpha)$ spaces has been extended to the Bessel potentials, to be applied further for solving the wave equations. In the third section we solve the wave equations associated with the Bessel differential operator, and we give some regularities results.

## 1. Preliminaries

Throughout this paper we fix $\alpha > -\frac{1}{2}$. First we remind some classical Harmonic Analysis results associated with the Bessel differential operator $\mathcal{L}_\alpha$ $(\alpha > -\frac{1}{2})$ given by

$$\mathcal{L}_\alpha = \frac{1}{x^{2\alpha+1}}\frac{d}{dx}(x^{2\alpha+1}\frac{d}{dx}). \tag{2}$$

The above operator satisfies some useful properties, like:

$$\mathcal{L}_\alpha(j_\alpha(\lambda x)) = -\lambda^2 j_\alpha(x), \quad \text{for all } \lambda \in \mathbb{R}. \tag{3}$$

Also, for all $f \in S_*(\mathbb{R})$, $\mathcal{F}_B(f)$ and $\mathcal{L}_\alpha(f)$ belong to $S_*(\mathbb{R})$, and we have

$$\mathcal{F}_B(\mathcal{L}_\alpha f)(\lambda) = -\lambda^2\mathcal{F}_B(f)(\lambda)\,;\ \mathcal{L}_\alpha[\mathcal{F}_B f](\lambda) = \mathcal{F}_B[(-x^2)f](\lambda) \text{ for all } \lambda \in \mathbb{R}. \tag{4}$$

These results are standard in the sense that they occur in the monographs on harmonic analysis, like for example [11]. In what follows, we equip $\mathbb{K}$ with the measure $dm_\alpha$ and $L^p(dm\alpha)$ with the norms

$$\|f\|_{L^p(dm_\alpha)} = \left(\int_0^\infty \mid f(x) \mid^p d(m_\alpha)\right)^{\frac{1}{p}}; \quad 1 \leq p < +\infty, \tag{5}$$

and

$$\|f\|_{L^\infty(dm_\alpha)} = ess \sup_{x\geq 0} \mid f(x) \mid.$$

The generalized translation operator $T_x^\alpha\,; x \in \mathbb{K}$ is defined for a suitable function $f$ and for all $y \in \mathbb{K}$ as follows ([11], p.93):

$$T_x^\alpha f(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi f\left(\sqrt{x^2+y^2+2xy\cos\theta}\right)(\sin\theta)^{2\alpha}d\theta. \tag{6}$$

These operators are well defined on $L^1(dm_\alpha)$, turning this space into a commutative Banach algebra and one has for all $x \in \mathbb{K}$, $\lambda \in \mathbb{C}$ ([11], p.13)

$$\left[T_x^\alpha j_\alpha(\lambda)\right](y) = j_\alpha(\lambda x) j_\alpha(\lambda y). \tag{7}$$

Furthermore, if $f$ belongs to $L^1(dm_\alpha)$, one has ([11], p.121)

$$\mathcal{F}_B(T_x^\alpha f)(\lambda) = j_\alpha(\lambda x)\mathcal{F}_B(f)(\lambda) \quad \text{for all } \lambda \in \mathbb{R} \text{ and } x \geq 0. \tag{8}$$

The generalized convolution product on $\mathbb{K}$ for appropriate pair of functions $f$ and $g$ is denoted $f\#g$ and is given by ([11], p.97)

$$(f\#g)(x) = \int_0^\infty T_x^\alpha f(y) g(y) dm_\alpha(y). \tag{9}$$

It has been proved in [11] that $(\mathbb{K}, \#)$ is a hypergroup in the sense of Jewett, called Bessel-Kingman hypergroup, and that the above convolution product satisfies the Young inequality, that is, for $f \in L^q(dm_\alpha)$ and $g \in L^p(dm_\alpha)$; $(p, q \in [1,\infty],\ \frac{1}{r}+1 = \frac{1}{q}+\frac{1}{p})$, $f\#g$ belongs to $L^r(dm_\alpha)$ ([11], p.100). Moreover, the Young inequality holds:

$$||f\#g||_{L^r(dm_\alpha)} \leq ||f||_{L^q(dm_\alpha)}.||g||_{L^p(dm_\alpha)}. \tag{10}$$

Meanwhile, for $f$ in $L^p(dm_\alpha)$ and $\psi \in S_*(\mathbb{R})$, $< f, \psi >$ means the value of $f \in S'_*(\mathbb{R})$ on $\psi$ and it is given by:

$$< f, \psi >= \int_0^\infty f(x)\psi(x) dm_\alpha(x). \tag{11}$$

It might be observed that a long list of properties of the classical distributions on $\mathbb{R}^n$ remains valid also in our context and that the operators $\mathcal{L}_\alpha$ and $\mathcal{F}$ can be extended naturally on $S'_*(\mathbb{R})$ as

$$< \mathcal{F}_B(T), \psi >=< T, \mathcal{F}_B(\psi) > \quad \text{and} \quad < \mathcal{L}_\alpha T, \psi >=< T, \mathcal{L}_\alpha\psi >, \tag{12}$$

for all $T \in S'_*(\mathbb{R})$ and $\psi \in S_*(\mathbb{R})$.

## 2. Bessel potential on the Bessel-Kingman hypergroups

In this section we investigate the Bessel potential on the Bessel-Kingman hypergroups $B_{\alpha,s}$, $(s \in \mathbb{R})$, to be the operator

$$B_{\alpha,s} = \mathcal{F}_B^{-1}\left[(1+\xi^2)^{-\frac{s}{2}}\mathcal{F}_B(.)\right].$$

One can remarks easily that $B_{\alpha,s} o B_{\alpha,t} = B_{\alpha,s+t}$ for any real $s$ and $t$ and that $B_{\alpha,0} = Id$.

The Bessel potential operator is a continuous isomorphism from $S_*(\mathbb{R})$ into itself. Its inverse is the operator $B_{\alpha,-s}$. And hence, it can be extended naturally on $S'_*(\mathbb{R})$ by means of (12).

Definition 2.1. For any real $s$ and $p \in [1,\infty]$, we define the space $W_\alpha^{s,p}$ to be the set of all tempered distributions $T$ such that $B_{\alpha,-s}(T)$ belongs to $L^p(dm_\alpha)$.

Using the fact that $B_{\alpha,-s}$ maps $S_*(\mathbb{R})$ into itself and that $S_*(\mathbb{R})$ is included in $L^p(dm_\alpha)$, one can prove easily that $S_*(\mathbb{R}) \subset W_\alpha^{s,p}$, for all $s \in \mathbb{R}$ and $p \in [1,\infty[$. Furthermore, $W_\alpha^{s,p}$ endowed with the norm $||.||_{W_\alpha^{s,p}} = ||B_{\alpha,-s}(.)||_{L^p(dm_\alpha)}$ are Banach spaces and $S_*(\mathbb{R})$ is dense in $\left(W_\alpha^{s,p}, ||.||_{W_\alpha^{s,p}}\right)$. On the other hand, it is clear that $W_\alpha^{s,2} = H_\alpha^s$ (see [3]) for all $s \in \mathbb{R}$, and that $B_{\alpha,t}\left(W_\alpha^{s,p}\right) = W_\alpha^{s+t,p}$. Moreover, $B_{\alpha,t}$ is an isometric isomorphism from $W_\alpha^{s,p}$ into $W_\alpha^{s+t,p}$. The name Bessel potential for these operators is related to the Fourier-Bessel transform of the symbol $(1+\xi^2)^{\frac{-s}{2}}$, that is ([5], p.235)

$$\mathcal{F}_B\left((1+\xi^2)^{\frac{-s}{2}}\right)(y)k_{\alpha,s}(y) = \frac{y^{\frac{s}{2}-1-\alpha}}{2^{\frac{s}{2}-1}}\frac{1}{\Gamma(\frac{s}{2})}K_{\alpha-\frac{s}{2}+1}(y), \tag{13}$$

where $K_\nu\,;(\nu \in \mathbb{C})$ is the MacDonald function defined on $\mathbb{C}\setminus(-\infty,0]$ by:

$$K_\nu(z) = \frac{I_{-\nu}(z) - I_\nu(z)}{\sin\nu\pi},$$

and $I_\nu$ is the modified Bessel function of order $\nu$ given by

$$I_\nu(z) = \sum_{k=0}^{\infty}\frac{1}{k!\Gamma(k+\nu+1)}\left(\frac{z}{2}\right)^{2k+\nu}\,; Arg(z) \in ]-\pi,\pi[.$$

The MacDonald function satisfies the following asymptotic behaviors (see [8], p.252):

- $K_\nu(x) \sim \sqrt{\frac{\pi}{2}}.\frac{e^{-x}}{\sqrt{x}} \qquad (x \to +\infty).$
- $K_\nu(x) \sim \frac{\Gamma(|\nu|)}{2}.\left(\frac{x}{2}\right)^{-|\nu|} \quad (x \to 0^+); \quad \nu \neq 0.$
- $K_0(x) \sim -Log(x) \qquad (x \to 0^+).$

Formula (13) is valid for all $s \in \mathbb{C}\backslash\{0,-2,-4,....,\}$. For $s=0,-2,-4,...,$ $-2k,\ k \in \mathbb{N}$, one has

$$k_{\alpha,s} = (1-\mathcal{L}_\alpha)^{-s}\delta.$$

The function $k_{\alpha,s}$ has to be understood as finite part in the sense of Hadamard for large $-s$.

EXAMPLE 2.1. *Considering the Dirac distribution $\delta$, we calculate the indexes $p$ and $s$ for which $\delta$ belongs to $W_\alpha^{s,p}$. For this, we need $B_{\alpha,s}(\delta)$ for arbitrary $s$.*

*Because supp$(\delta) = \{0\}$ is compact, we may use $k_{\alpha,s}$ and the representation*

$$B_{\alpha,s}(\delta) = k_{\alpha,s}\#\delta = k_{\alpha,s}.$$

*From this, and the behavior of $k_{\alpha,s}$ we get*

$$k_{\alpha,s} \in L^p(dm_\alpha) \text{ for } s > \left(1-\frac{1}{p}\right)2(\alpha+1).$$

For working with convolution and Bessel potential, the following theorem is useful.

THEOREM 2.1. *Let $s \in \mathbb{R}$ and $f, g \in S'_*(\mathbb{R})$ be convoluvable in $S'_*(\mathbb{R})$ (in the sense that $f\#g$ and $g\#f$ exist and satisfy $f\#g = g\#f$), then we get*

$$B_{\alpha,s}(f\#g) = B_{\alpha,s}(f)\#g = f\#B_{\alpha,s}(g). \tag{14}$$

P r o o f. Let $f, g$ in $S'_*(\mathbb{R})$ being convoluvable. Using the fact that the translation operator commutes with the Bessel potential, we obtain

$$< B_{\alpha,s}(f\#g), \varphi >=< f\#g, B_{\alpha,s}\varphi >$$

$$=< f \otimes g, T_x^\alpha(B_{\alpha,s}\varphi(y)) >=< f \otimes g, B_{\alpha,s}(T_x^\alpha\varphi)(y) >$$

$$=< f, < g, B_{\alpha,s}(T_x^\alpha\varphi)(y) >>=< f, < B_{\alpha,s}(g), (T_x^\alpha\varphi)(y) >>$$

$$=< f \otimes B_{\alpha,s}(g), (T_x^\alpha\varphi)(y) >=< f\#B_{\alpha,s}(g), \varphi > .$$

This proves the first equality. The rest of the proof goes as before. ∎

EXAMPLE 2.2. *The Heaviside function*

$$H(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1 & \text{if} \quad x > 0. \end{cases}$$

*belongs to $W_\alpha^{s,\infty}$ for all $s \geq 0$.*

Taking into account that $B_{\alpha,s}(H) = B_{\alpha,s}(H\#\delta) = H\#B_{\alpha,s}(\delta) = k_{\alpha,s}\#H$, it follows that $B_{\alpha,s}(H)$ is bounded whenever $k_{\alpha,s} \in L^1(dm_\alpha)$ or $k_{\alpha,s} = \delta$. This is true for all $s \geq 0$.

The theorem of Young on convolution in $L^p(dm_\alpha)$ spaces can be extended immediately to the spaces $W_\alpha^{s,p}$, as follows.

THEOREM 2.2. *Let $p, q \in [1, +\infty]$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0, s, s' \in \mathbb{R}$. If $f \in W_\alpha^{s,p}$ and $g \in W_\alpha^{s',q}$, then the distributions $f$ and $g$ are convoluvable, and $f\#g$ belongs to $W_\alpha^{s+s',r}$ and satisfies*

$$\|f\#g\|_{W_\alpha^{s+s',r}} \leq \|f\|_{W_\alpha^{s,p}} \|g\|_{W_\alpha^{s',q}}.$$

P r o o f. Using the semi-group properties of the Bessel potential operators, the property (10), and Theorem 2.1, one can write

$$\|f\#g\|_{W_\alpha^{s+s',r}} = \|B_{\alpha,(-s-s')}f\#g\|_{L^r}$$
$$= \|B_{\alpha,(-s)}f\#B_{\alpha,(-s')}g\|_{L^r} \leq \|f\|_{W_\alpha^{s,p}} \|g\|_{W_\alpha^{s',q}}.$$

This ends the proof. ■

As an application of the above theorem, we give the following Sobolev imbedding result.

COROLLARY 2.1. *For $p \leq q$ and $s' < s + \left(\frac{1}{q} - \frac{1}{p}\right)(2\alpha + 2)$, the following continuous imbedding holds:*

$$W_\alpha^{s,p} \hookrightarrow W_\alpha^{s',q}.$$

P r o o f. The imbedding is given by convolution with $\delta$. We know that $\delta \in W_\alpha^{r,\gamma}$ as soon as $\gamma > \left(1 - \frac{1}{r}\right)(2\alpha + 2)$ (see Example 2.1). And from the above theorem, with $\frac{1}{r} - 1 = \frac{1}{q} - \frac{1}{p}$ and $s - s' = \gamma > \left(1 - \frac{1}{r}\right)(2\alpha + 2)$, we get

$$\|f\|_{W_\alpha^{s',q}} \leq \|f\|_{W_\alpha^{s,p}} \|\delta\|_{W_\alpha^{s-s',r}}.$$

Moreover, for $r = 1$, that is $p = q$ and $s > s'$, we obtain

$$\|f\|_{W_\alpha^{s',p}} \leq C_{s,s',\alpha} \|f\|_{W_\alpha^{s,p}}, \tag{15}$$

where $C_{s,s',\alpha} = \|\delta\|_{W_\alpha^{s-s',1}}$. ■

The embedding given in inequality (15) was proved by Guliev and Safarov for the multidimensional case, see [6].

### 3. Applications: The wave equations

The aim of this section is to obtain an exact representation of solutions for the Cauchy problem assigned to the hyperbolic operator

$$\Box_\alpha = \partial_t^2 - \mathcal{L}_\alpha.$$

Let $s, s' \in \mathbb{R}$ and $p, q, r \geq 1$. We consider the generalized wave equation assigned to the Bessel differential operator and we look for weak solutions of

$$(P_w)\begin{cases} \Box_\alpha\, u(x,t) = 0 \\ \\ u(x,0) = \Phi \in W_\alpha^{s,p} \\ \\ \frac{\partial u}{\partial t}(x,0) = \Psi \in W_\alpha^{s',q}, \end{cases}$$

belonging for fixed time $t$ to $S'_*(\mathbb{R})$, i.e. $u \in \mathcal{C}^2([0,\infty), S'_*(\mathbb{R}))$, the data $\Phi$ and $\Psi$ are being respectively in $W_\alpha^{s,p}$ and $W_\alpha^{s',q}$. Under these assumptions, it is possible to apply the partial Fourier-Bessel transform to the above Cauchy problem to obtain:

$$u(x,t) = B_{\alpha,s}(f)\#\chi_t(x) + B_{\alpha,s'}(g)\#\psi_t(x), \tag{16}$$

where $f = B_{\alpha,-s}(\Phi)$, $g = B_{\alpha,-s'}(\Psi)$ and $\chi$ and $\psi$ are such that $\mathcal{F}_B(\chi_t)(x) = \cos xt$ and $\mathcal{F}_B(\psi_t)(x) = \dfrac{\sin xt}{x}$. The convolution in (16) is well defined, since $\cos(t\xi)$ and $\dfrac{\sin(t\xi)}{\xi}$ belong to $\mathcal{H}_*(\mathbb{C})$ (see [1]) and the Fourier Bessel transform is an isomorphism from the space $\mathcal{E}'_*(\mathbb{R})$ onto $\mathcal{H}_*(\mathbb{C})$ (see [11], p.146).

In what follows, we consider $a$ and $b \in \mathbb{R}$ such that $\chi_t \in W_\alpha^{a,2}$ and $\psi_t \in W_\alpha^{b,2}$; that is $a > \frac{1}{2}$ and $b > -1$. Under these conditions and using Theorem 2.2, we give the regularity of solutions for the problem $(P_w)$ by means of the data $\Phi$ and $\Psi$:

- For $p = q = 2$ and $a + s = b + s' = c$, the solution $u$ belongs to $W_\alpha^{a+s,\infty} + W_\alpha^{b+s',\infty}$. Moreover the following estimate follows

$$||u||_{W_\alpha^{c,\infty}} \leq c(||\Phi||_{W_\alpha^{s,2}} + ||\Psi||_{W_\alpha^{s',2}}).$$

[1] $\mathcal{H}_*(\mathbb{C}) = \bigcup_{a\geq 0} \mathcal{H}_{*,a}(\mathbb{C})$, where $\mathcal{H}_{*,a}(\mathbb{C})$ is the space of even entire functions $\varphi$ on $\mathbb{C}$, slowly increasing and of exponential type $a$, that is $\exists m \in \mathbb{N}, q_m(\varphi) = \sup_{\lambda\in\mathbb{C}} \left(1+|\lambda|^2\right)^{-m} |\varphi(\lambda)| e^{-a|Im\lambda|} < +\infty$

- For $p \neq 2$ and $q \neq 2$ we have $u \in W_\alpha^{a+s,\frac{2p}{2-p}} + W_\alpha^{b+s',\frac{2q}{2-q}}$. And for $a+s = s'+b = c$ and $p = q$ we get

$$||u||_{W_\alpha^{c,\frac{2p}{2-p}}} \leq c(||\Phi||_{W_\alpha^{s,p}} + ||\Psi||_{W_\alpha^{s',p}}).$$

Now, for $\Phi$ and $\Psi$ in $S_*(\mathbb{R})$ the solution of the problem $(P_w)$ belongs also to $S_*(\mathbb{R})$ and the principle of conservation of energy is that

$$\int_0^\infty \left( u_t^2 + \left(\frac{\partial u}{\partial x}\right)^2 \right) x^{2\alpha+1} dx$$

is independent of the time $t$. And for $s = s'$ we have

$$||u||^2_{D^s} := ||u_t||^2_{H^s_\alpha} + ||\mathcal{L}_\alpha^{\frac{1}{2}} u||^2_{H^s_\alpha}$$

is constant with respect to time. Moreover,

$$||u_t||^2_{H^s_\alpha} + ||\mathcal{L}_\alpha^{\frac{1}{2}} u||^2_{H^s_\alpha} = ||\mathcal{L}_\alpha^{\frac{1}{2}} f||^2_{L^2(dm_\alpha)} + ||g||^2_{L^2(dm_\alpha)},$$

where $\mathcal{L}_\alpha^s$ ; $s \in \mathbb{R}$ is the fractional power of $\mathcal{L}_\alpha$.

## References


[1] R. Adams, *Sobolev Spaces.* Academic Press, New York - San Francisco - London (1975).

[2] I.A. Aliev and M. Eryigit, Inversion of Bessel potentials with the aid of weighted wavelet transforms. *Math. Nachrichten* **242** (2002), 27-37.

[3] M. Assal, M. Nessibi, Bessel-Sobolev type spaces. *Mathematica Balkanica (N.S.)* **18**, Fasc. 3-4 (2004), 227-234.

[4] D.I. Cruz-Baez and J. Rodriguez, Spaces of distributions of Besov and Lizorkin-Triebel type for the Fourier-Bessel transform. *J. of Math. Analysis and Applications* **259** (2001), 51-63.

[5] A. Erdélyi (Ed.), *Tables of Integral Transforms*, Vol II. McGraw-Hill, New York (1954).

[6] V.S. Guliev and Z.V. Safarov, $B_{k,n}$-Bessel potentials and certain imbed- ding theorems in $B_{k,n}$-Sobolev-Liouville spaces. *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **15** (2001), 68-80.

[7] N.M. Lebedev, *Special Functions and Their Applications.* Dover, New York(1972).

[8] F.W.J. Olver, *Asymptotics and Special Functions.* Academic Press, New York (1974).

[9] R.S. Pathak and P.K. Pandey, Sobolev type spaces associated with Bessel operators. *J. of Math. Analysis and Applications* **215** (1997), 95-111.

[10] L. Schwartz, *Théorie des disrtibutions*, Tomes I et II. Hermann, Paris (1957).

[11] K. Trimèche, *Generalized Harmonic Analysis and Wavelet Packets*. Gordon and Breach Sci. Publ., The Netherlands (2000).

[12] G.N. Watson, *A Treatise of the Theory of Bessel Functions*. Second Ed., Cambridge University Press, London - New York (1966).

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